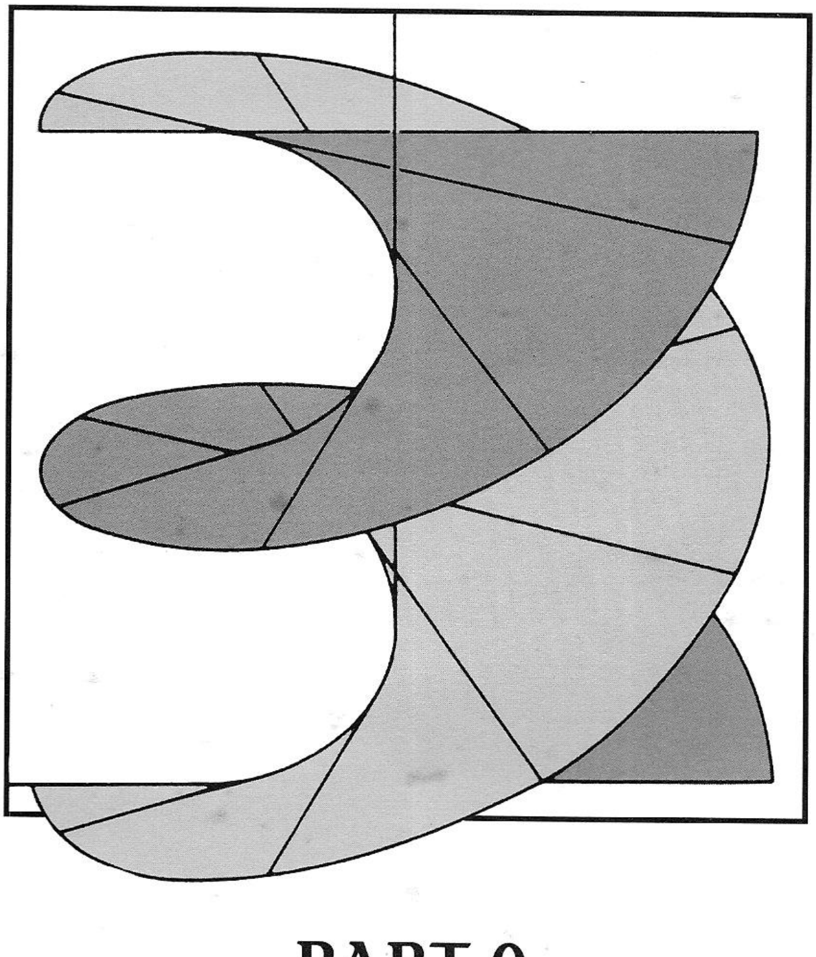
M434 Part 0
Mathematics: A Fourth Level Course

M434
DIFFERENTIAL GEOMETRY



PART 0
COURSE GUIDE AND INTRODUCTION

The Open University

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Mathematics: A Fourth Level Course

M434 Differential Geometry

Part 0 Course Guide and Introduction

Prepared for the Course Team
by Bob Margolis

Set book

Barrett O'Neill, *Elementary Differential Geometry*, hardback edition (Academic Press, 1966).

It is essential to have this book; the course is based on it and will not make sense without it.

The set book is referred to as *O'Neill*.

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This text forms part of an Open University Fourth Level Course. If you would like a copy of *Studying with The Open University*, please write to the Central Enquiry Service, PO Box 200, The Open University, Walton Hall, Milton Keynes, MK7 6YZ. If you have not already enrolled on the Course and would like to buy this or other Open University material, please write to Open University Educational Enterprises Ltd, 12 Cofferidge Close, Stony Stratford, Milton Keynes, MK11 1BY, United Kingdom.

1.2

Contents

Introduction

This text provides a guide to the course. It also contains some mathematical material which provides a link between what you have probably done already and the beginning of the set book, *O'Neill*.

It is important that you should read right through this text to check on the conventions used and the purpose of the various components of the course material.

The first section is a guide to the various components of the course, plus a brief historical note about the development of the ideas that appear in the course.

In Section 2, we give a very brief survey of the main mathematical prerequisites. There are a few exercises to remind you of the most important techniques that you will need.

1 Course guide

This course provides an introduction to the ideas and methods of differential geometry by studying curves and surfaces in three-dimensional space. It deliberately restricts its attention to three dimensions in order to enable close links to be kept with your intuitive ideas about geometry. Nevertheless, the ideas and basic techniques developed in this course do generalize and will provide you with a good foundation should you wish to read further in the subject.

1.1 Components of the course

The course consists of four main components.

- The set book.
- The course texts, of which this document is the first.
- The Tutor-marked Assignments (TMAs).
- The examination.

We shall look at each of these in turn.

The set book

The main source of mathematical material for the course is the set book:

Barrett O'Neill, *Elementary Differential Geometry* (Academic Press, 1966).

In future, the set book will be referred to simply as *O'Neill*.

It is essential to have this book as the course material will make no sense without it. The mathematical scope of the course is almost entirely defined by the sections of *O'Neill* that we have included.

As a half-credit course cannot cover all the material in *O'Neill*, we have selected the material to cover the geometric ideas in three dimensions and also to give you the tools with which to study further if you wish.

Course texts

The printed texts, labelled *Part 0* and *Parts I–VI*, are intended to provide a link between previous Open University mathematics courses that you may have studied and the set book. They also contain some additional teaching material, suggest exercises for you to do to reinforce your understanding and provide solutions to those exercises. Each of *Parts I–VI* contains material associated with the corresponding chapter of *O'Neill*. The remainder of this text contains material linking second-level mathematics courses with *M434*.

We shall refer to these simply as 'texts'.

Tutor-marked Assignments

There are four Tutor-marked Assignments that, together, make up the continuous assessment component of the course. Up to one of the scores on these TMAs may be substituted in accordance with University regulations. All four assignments are equally weighted.

The assignments are mostly concerned with applying the techniques developed in the course to particular examples of curves, surfaces, etc. However, where appropriate, a question may be included that extends the course work in some measure.

The examination

There is a three-hour final examination for the course. Your course result will be determined from your continuous assessment score and examination score using equal weighting.

The examination is intended to be a straightforward test of the basic ideas of the course. Because we wish to be able to test small, but central, pieces of 'bookwork', it is a closed-book examination. We want to reassure you that this does *not* mean that the examination will be a memory test! We simply want to be able to test your grasp of the central ideas and your ability to apply them to straightforward cases, even if they have appeared in the course material; this we could not do if a handbook were permitted in the examination.

Tutorials and regional support

Your Regional Centre will inform you of the name and address of your tutor, to whom you will send your assignments. They will also send you details of any tutorials, day schools or other forms of contact that have been arranged in your Region.

As this course has a small number of students, it is unlikely that there will be very much face-to-face tuition but the Region may arrange some other form of contact with your tutor.

1.2 The course structure

The main aims of the course could be stated as:

- to extend the ideas of elementary calculus to calculus in three dimensions and on surfaces;
- to develop mathematical tools useful for the study of curves and surfaces;
- to give precise mathematical formulations of intuitive notions about the 'shapes' of curves and surfaces;
- to develop computational techniques for investigating the properties of particular curves and surfaces.

The four blocks

The course breaks down into four blocks, each corresponding to approximately four units' work (in the conventional OU sense). These four blocks correspond to the TMAs and the aims discussed above in the following way.

Assignment	Sections of *O'Neill*	Content
TMA M434 01	Chapter I, 1–9 and Chapter II, 1–2	*Mathematical Tools* and methods required for the rest of the course
TMA M434 02	Chapter II, 3–9 and Chapter III, 1–6	*Curves* in three dimensions
TMA M434 03	Chapter IV, 1–5 and Chapter V, 1–3	*Introduction to Surfaces* in three dimensions
TMA M434 04	Chapter V, 4–7 and Chapter VI, 1, 2 and 6	*Further Study of Surfaces*

It is intended that each section of *O'Neill* should be studied in conjunction with the corresponding text.

The four blocks above vary somewhat in nature and difficulty. The structure of the blocks is imposed upon us by the structure of *O'Neill*; it is, perhaps, not what we would have chosen if we had had a totally free hand. The comments on individual blocks below may help to prepare you for some of the difficulties.

- The *Mathematical Tools* block is, by its nature, rather abstract and gives little indication of how the tools are to be used for the study of curves and surfaces. The introductory mathematical material in this text and the commentary in *Part I* are intended to help with this problem and to make it somewhat easier to organize a rather confusing mass of material.
- The *Curves* block is much more concrete than the first one. Some of the tools of *Part I* are applied to studying curves in three dimensions, and powerful methods of describing curves are obtained. In spite of the power of the results obtained, the methods used are fairly straightforward applications of calculus and linear algebra.
- The *Introduction to Surfaces* block is a mixture of some rather intricate definitions and straightforward calculation techniques. One of the main aims is to prove some results that show that, although careful definition of what a surface *is* is necessary, surfaces can be *described* rather more simply.
- The *Further Study of Surfaces* block develops computational techniques for describing the shapes of surfaces, based on the tools from the first block.

1.3 Study

Although you may well have gained considerable experience in studying OU or other courses our suggestions will make the assumption that your experience of study using set books may be more limited.

The TMA-defined blocks are the smallest coherent components of *M434*. Within these blocks, the sections into which the chapters of *O'Neill* are divided provide smaller pieces for study sessions. Unfortunately, the sections of *O'Neill* vary quite a lot in required study time; the texts try to give a rough indication of how they compare to a 'standard' study week of 12–14 hours.

We would suggest that you start the study of a block by skimming through the text. The text will provide an introduction to the block, define the sections of *O'Neill* that are included in the block, give an estimate of relative study times, suggest

exercises, provide solutions to exercises and provide commentary on *O'Neill*. Each section of the text contains, when appropriate, a note of the errata that we have found in *O'Neill*. You should check these before reading the section of *O'Neill*. Whilst on the subject of errata, please note that several reprints, with corrections, of *O'Neill* have appeared over the years. We have tried to give all errata for all printings; some of those given may not, therefore, apply to your copy.

It is probably sensible to tackle the reading from *O'Neill* next, one section at a time. Whether you find that you can work straight through *O'Neill*, or you find that you need to consult the commentary, will depend very much on you. However you do it, please be sure to read the commentary in the accompanying text too. Sometimes *O'Neill* puts an important result in an exercise. The text will direct you to do such exercises (as well as others) and also draw particular attention to the result. It may be that there is some additional material that we feel is necessary, as well as help with the concepts in *O'Neill*.

1.4 Historical note

This note is intended to give a brief historical perspective to some of the aspects of differential geometry that are dealt with in this course. Differential geometry is a vast subject area and most of the great mathematicians of the last three centuries have made some contribution to it.

The study of curves in the plane started well before Newton and Leibniz laid the foundations for modern calculus. Huygens (1629–1695) studied curves in the course of his work on light. However, the calculus enabled Newton (1647–1727) to make a much more detailed study of plane curves, and Euler (1707–1783) studied curves on surfaces more general than the plane. The study of surfaces themselves was opened up by the use of 'patches' which were introduced (in a restricted way) by Monge (1746–1818) and then used by Gauss (1777–1855) to describe surfaces in an intrinsic way. This study of surfaces provided the answers to a number of problems including whether it is possible to make a plane map of the globe that retains all the 'local' properties of the geometry of the globe. (It is not!)

It was Riemann (1826–1866) who saw how the theory of surfaces could be generalized. The study of Riemann surfaces showed that Euclidean geometry was not the only possibility. There is now a vast literature on non-Euclidean geometries.

Meanwhile, the theory of curves took a great step forward with the discovery by Frenet (1816–1900) and Serret (1819–1885) of formulas describing the curvature and torsion of curves in three-dimensional space. Their method involved 'moving frames' attached to the curve. This approach was extended to surfaces by Darboux (1842–1907) and generalized fully by Cartan (1869–1951) who brought back geometric insight to the profusion of equations produced by Riemann and his followers.

2 Mathematical prerequisites

It is assumed that you have met before a number of mathematical techniques and ideas that are used in the course. We discuss these in this section, indicating the OU courses where you may have met them, and offer a review of the most important techniques together with some practice examples.

In writing this course we have assumed that you have studied one or preferably both of the following courses:

(a) *M203 Introduction to Pure Mathematics*;

(b) *MST204 Mathematical Models and Methods.*

The courses will, from now on, simply be referred to by their course codes.

These courses represent the OU source for the prerequisite mathematics, but all that matters is that you have met the necessary ideas *somewhere*.

Most of the required mathematics is provided by *M203*, but some ideas appear only in *MST204*. If you have the Handbooks for either (or both) of these courses, you will find them useful for reference. You may also find the *Mathematics Foundation Course Handbook* useful.

The prerequisites can be divided into the following three areas:

- calculus;
- vector methods;
- linear algebra.

In each area you will need both the basic ideas and manipulative skill with techniques. We shall look at each area in turn.

2.1 Calculus

You will do a lot of differentiating whilst studying *M434*! You are expected to know the derivatives of many standard functions: polynomials, the trigonometric functions, e^x, $\log_e x$ and so on. You are also expected to know, and be able to use, the various rules for differentiating complicated functions by breaking them down into smaller pieces.

M203 Handbook

Specifically, for functions f and g from $\mathbf{R}$ to $\mathbf{R}$, you should know how to use the rules:

$$(af+bg)' = af' + bg' \quad (a, b \in \mathbf{R}) \quad \text{linearity;}$$
$$(fg)' = f'g + fg' \quad \text{product rule or Leibniz property;}$$
$$(f \circ g)' = f'(g)g' \quad \text{composite function or chain rule.}$$

Perhaps one of the most important things to realize about these rules is that they are *theorems* that follow from the definition of what is meant by a derivative. The theorems are needed so that, once a list of basic derivatives has been obtained (perhaps from first principles), more complicated functions can be differentiated by dissecting the function until it has been expressed in terms of functions in the basic list.

A simple example may illustrate the point. Suppose

$$h(x) = \sin(\cos x + x^2),$$

then

$$\begin{aligned} h'(x) &= (\sin'(\cos x + x^2)) \times (\cos x + x^2)' && \text{chain rule} \\ &= (\sin'(\cos x + x^2)) \times ((\cos x)' + (x^2)') && \text{linearity} \\ &= (\cos(\cos x + x^2)) \times (-\sin x + 2x) && \text{basic derivatives} \\ &= (-\sin x + 2x)\cos(\cos x + x^2). \end{aligned}$$

You may find the above a bit long-winded, nevertheless it is what you actually do when finding $h'(x)$ even if you do not write down all the stages.

You will meet many forms of derivative in the course but all will have the linearity and Leibniz (product) properties and (where appropriate) a chain rule as well. Usually the first activity that *O'Neill* does after defining a new derivative is to prove that it has these properties. Equally, the technique for finding the derivative of a complicated function will still be the method of dissection until it is expressed in terms of a few basic derivatives.

It is assumed in this course that you are familiar with *partial differentiation*, which is discussed in *MST204*, but not in *M203*. What follows is intended to be a brief reminder of the ideas and notation of partial differentiation.

Partial derivatives are associated with 'functions of several variables' or, if you prefer, functions from $\mathbf{R}^2$ to $\mathbf{R}$ and $\mathbf{R}^3$ to $\mathbf{R}$. Examples of such functions are

$$f(x,y) = x^2 + xy^2 - \sin(xy) \quad \text{and} \quad g(x,y,z) = xy + x^2z - 2xy^2z.$$

If we regard all variables except one as constants and differentiate with respect to the remaining variable, we obtain a partial derivative with respect to that variable. For example, using the function f above, the partial derivative with respect to x is obtained by treating y as a constant and is

$$2x + y^2 - y\cos(xy).$$

The first term is just the derivative of x^2, the second is the derivative of x times the 'constant' y^2, and the last term was obtained by applying the chain rule to $\sin(xy)$.

Exercise 2.1 Find the partial derivative of f, above, with respect to y.

[*Solution on page 15*]

To distinguish partial derivatives from the usual (one-variable) derivatives, we use

$$\frac{\partial f}{\partial x}$$

for the partial derivative of f with respect to x. The symbol $\partial f/\partial x$ is read

'partial dee f by dee x'.

Thus, the answer to the last exercise can be written

$$\frac{\partial f}{\partial y} = 2xy - x\cos(xy).$$

Exercise 2.2 For the function g, defined above, calculate

$$\frac{\partial g}{\partial x}, \quad \frac{\partial g}{\partial y} \quad \text{and} \quad \frac{\partial g}{\partial z}.$$

[*Solution on page 15*]

Finally, in *M434* we assume familiarity with the 'hyperbolic' functions defined by

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

$$\sinh x = \frac{e^x - e^{-x}}{2},$$

$$\tanh x = \frac{\sinh x}{\cosh x}.$$

You have probably met these before but we suspect that you have had less practice manipulating them than, for example, the usual trigonometric functions.

Most of the properties of the hyperbolic functions that we shall need can be derived from the above definitions.

Exercise 2.3 Show that

(a) $\cosh^2 x - \sinh^2 x = 1$;

(b) $\operatorname{sech}^2 x = 1 - \tanh^2 x$, where $\operatorname{sech} x = 1/\cosh x$.

Exercise 2.4 Show that

(a) $\cosh' x = \sinh x$;

(b) $\sinh' x = \cosh x$;

(c) $\tanh' x = \operatorname{sech}^2 x$.

[*Solutions on page 15*]

Occasionally, we shall need the inverse functions of $\sinh x$ and $\cosh x$. Actually, only $\sinh x$ has an inverse without restricting the domain. We can obtain an inverse for $\cosh x$ if we restrict the domain to the non-negative reals.

To obtain an explicit formula for the inverse of $\sinh x$, we can proceed as follows.

Suppose that

$$y = \sinh x,$$

then

$$y = \frac{e^x - e^{-x}}{2}$$

and so

$$2y = e^x - e^{-x}.$$

Multiplying by e^x and rearranging, we obtain a quadratic in e^x:

$$(e^x)^2 - 2ye^x - 1 = 0.$$

Application of the quadratic equation formula gives

$$e^x = \frac{2y + \sqrt{4y^2 + 4}}{2} = y + \sqrt{y^2 + 1}.$$

Note that we cannot have a minus sign in the numerator, because

$$\sqrt{y^2 + 1} > y$$

and e^x is always positive.

Taking logs, we have

$$x = \log_e(y + \sqrt{y^2 + 1}).$$

Thus,

$$\sinh^{-1} : x \longmapsto \log_e(x + \sqrt{x^2 + 1}).$$

Similar calculations give

$$\cosh^{-1} : x \longmapsto \log_e(x + \sqrt{x^2 - 1}).$$

This function is the inverse of

$$\cosh : x \longmapsto \cosh x, \quad x \geq 0.$$

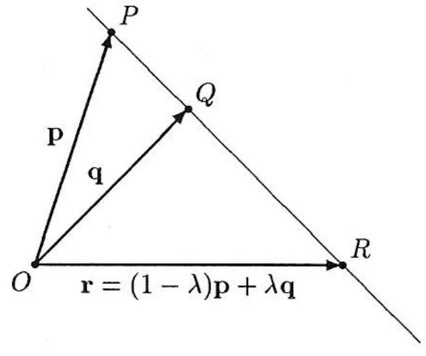
2.2 Vector methods

You will need to be familiar with some of the ways of using vectors to tackle geometric problems. The idea that appears most frequently in *M434* is that of representing a line (in $\mathbf{R}^3$) in vector form.

In what follows we use the common practice of representing a point named by a given capital letter, P say, by the vector from the origin to P, which we label with the corresponding bold lower-case letter, $\mathbf{p}$.

In *M203*, you met a way of representing the line joining the points P and Q in vector form as

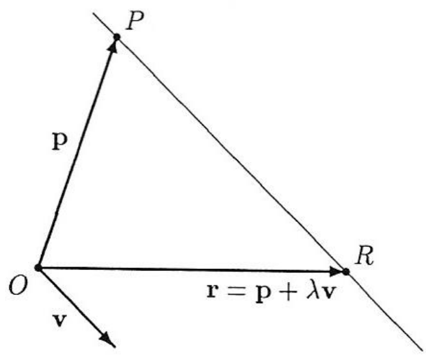
$$\mathbf{r} = (1 - \lambda)\mathbf{p} + \lambda\mathbf{q}, \quad \lambda \in \mathbf{R}.$$



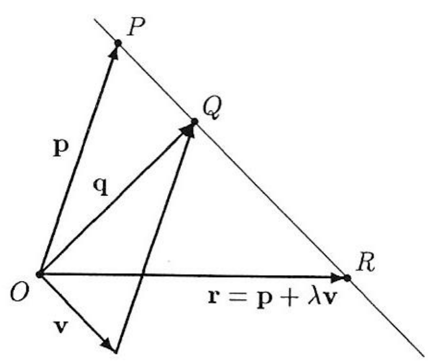
This equation is intended to mean that a general point R on PQ can be written in the form $(1-\lambda)\mathbf{p}+\lambda\mathbf{q}$ for some real number λ and, if we take all real values for λ, we obtain all the points on PQ.

In *M434* we take a slightly different approach, defining a line by giving a point P on it and the direction in which the line points. We define the direction by a vector, $\mathbf{v}$ say. The vector equation of the line becomes

$$\mathbf{r}=\mathbf{p}+\lambda\mathbf{v}, \quad \lambda \in \mathbf{R}.$$



These two forms of description of a line are really the same, as we can see if we combine the two diagrams above and do a little algebra.



From the diagram above, it should be clear that

$$\mathbf{q}=\mathbf{p}+\mathbf{v}$$

so

$$\begin{aligned}\mathbf{r}&=(1-\lambda)\mathbf{p}+\lambda\mathbf{q}\\&=(1-\lambda)\mathbf{p}+\lambda(\mathbf{p}+\mathbf{v})\\&=\mathbf{p}-\lambda\mathbf{p}+\lambda\mathbf{p}+\lambda\mathbf{v}\\&=\mathbf{p}+\lambda\mathbf{v}.\end{aligned}$$

Thus the 'two points' and 'point plus direction' forms are just two ways of saying the same thing. They correspond exactly to the 'two points' and 'one point plus slope' forms of the equation of a straight line in two-dimensional coordinate geometry. The relation

$$\mathbf{q}=\mathbf{p}+\mathbf{v}$$

allows us to switch between the two forms if necessary.

Exercise 2.5 Suppose that

$$\mathbf{p} = (1, 2, -3) \quad \text{and} \quad \mathbf{q} = (3, 1, 4).$$

(a) Write down the vector equation of the line PQ.

(b) Find a suitable vector $\mathbf{v}$ so that the equation of PQ can be expressed in the form $\mathbf{p} + \lambda\mathbf{v}$.

[*Solution on page 15*]

2.3 Linear algebra

We shall require a number of techniques connected with vector spaces and linear transformations between vector spaces and matrices.

All the ideas required are discussed in *M203*.

In what follows, we shall use examples from $\mathbf{R}^3$, because that, together with $\mathbf{R}^2$, appears most often in the course. Most of the ideas generalize quite naturally to $\mathbf{R}^n$.

Bases

A central concept in vector spaces is that of a basis: a set of vectors with the property that every vector in the space can be expressed uniquely as a linear combination of the basis vectors. For much work that you have done before, you probably used the 'standard' basis in $\mathbf{R}^3$:

$$\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

Although we shall often use this standard basis, others will play an important role as well. However, where possible, we shall try to use bases that have two important features in common with the standard basis. The vectors of the standard basis are

- unit length,
- mutually perpendicular.

To justify these statements requires a reminder of another idea: that of dot product. The dot product of two vectors in $\mathbf{R}^3$

$$\mathbf{p} = (p_1, p_2, p_3) \quad \text{and} \quad \mathbf{q} = (q_1, q_2, q_3)$$

is defined as

$$\mathbf{p} \cdot \mathbf{q} = p_1q_1 + p_2q_2 + p_3q_3.$$

The length of a vector can be expressed easily in terms of the dot product as

$$\|\mathbf{p}\| = \sqrt{\mathbf{p} \cdot \mathbf{p}}.$$

We use $\|\mathbf{p}\|$ for the length of a vector, rather than $|\mathbf{p}|$, in this course.

You may have seen the dot product defined as the product of the lengths and the cosine of the angle between the vectors, which is equivalent to our definition. The important property (for now) of the dot product is that the dot product of non-zero vectors is zero if, and only if, they are perpendicular.

It follows from the above discussion (by very easy calculations) that the vectors in the standard basis are unit length and mutually perpendicular.

The importance for us of using a basis of orthogonal unit vectors is in the ease with which it is possible to express an arbitrary vector as a linear combination of such basis vectors.

> **Orthonormal basis**
>
> A basis consisting of mutually orthogonal unit vectors is called an *orthonormal basis.*

Exercise 2.6 Suppose that $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is an orthonormal basis of $\mathbf{R}^3$ and $\mathbf{v}$ is a vector in $\mathbf{R}^3$. Assume that

$$\mathbf{v} = \alpha\mathbf{e}_1 + \beta\mathbf{e}_2 + \gamma\mathbf{e}_3, \quad \alpha, \beta, \gamma \in \mathbf{R}.$$

Show that

$$\alpha = \mathbf{v} \cdot \mathbf{e}_1, \quad \beta = \mathbf{v} \cdot \mathbf{e}_2, \quad \gamma = \mathbf{v} \cdot \mathbf{e}_3.$$

[*Solution on page 15*]

The result of the last exercise will be a very useful tool in the course. It is worth restating it as a general principle for $\mathbf{R}^n$.

Orthonormal expansion

If $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ is an orthonormal basis of $\mathbf{R}^n$ and $\mathbf{v}$ is a vector in $\mathbf{R}^n$, then

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1 + (\mathbf{v} \cdot \mathbf{e}_2)\mathbf{e}_2 + \cdots + (\mathbf{v} \cdot \mathbf{e}_n)\mathbf{e}_n.$$

This expansion is called the *orthonormal expansion* of $\mathbf{v}$ with respect to the basis.

Linear transformations

You have met the idea of a linear transformation from one vector space to another. *M203* We shall deal with only the cases where domain and codomain are of dimension up to three. That is, we consider linear transformations

$$\phi : \mathbf{R}^m \longrightarrow \mathbf{R}^n,$$

where $1 \leq m, n \leq 3$.

As a reminder, the term *linear* means that

$$\phi(a\mathbf{v} + b\mathbf{w}) = a\phi(\mathbf{v}) + b\phi(\mathbf{w}),$$

for all vectors $\mathbf{v}$ and $\mathbf{w}$ in the domain and all real numbers a and b.

By choosing a basis for each of the domain and codomain, we can associate a matrix with a linear transformation. As an example, suppose

$$\phi : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$$

is linear. Suppose that

$$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$$

is a basis for the domain and that

$$\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$$

is a basis for the codomain. The images of the basis vectors $\mathbf{e}_i$ can be expressed as linear combinations of the codomain basis vectors:

$$\begin{aligned}
\phi(\mathbf{e}_1) &= a_{11}\mathbf{f}_1 + a_{12}\mathbf{f}_2 + a_{13}\mathbf{f}_3,\\
\phi(\mathbf{e}_2) &= a_{21}\mathbf{f}_1 + a_{22}\mathbf{f}_2 + a_{23}\mathbf{f}_3,\\
\phi(\mathbf{e}_3) &= a_{31}\mathbf{f}_1 + a_{32}\mathbf{f}_2 + a_{33}\mathbf{f}_3.
\end{aligned}$$

The matrix

$$\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

is the matrix of ϕ with respect to the two bases. Note the order of the coefficients. It is correct for the following reason.

Suppose that the coordinates of $\mathbf{v}$ with respect to the basis in the domain are (a, b, c), that is

$$\mathbf{v} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3.$$

Then, using linearity,

$$\begin{aligned}
\phi(\mathbf{v}) &= \phi(a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3) \\
&= a\phi(\mathbf{e}_1) + b\phi(\mathbf{e}_2) + c\phi(\mathbf{e}_3) \\
&= a(a_{11}\mathbf{f}_1 + a_{12}\mathbf{f}_2 + a_{13}\mathbf{f}_3) + b(a_{21}\mathbf{f}_1 + a_{22}\mathbf{f}_2 + a_{23}\mathbf{f}_3) \\
&\quad + c(a_{31}\mathbf{f}_1 + a_{32}\mathbf{f}_2 + a_{33}\mathbf{f}_3) \\
&= (aa_{11} + ba_{21} + ca_{31})\mathbf{f}_1 + (aa_{12} + ba_{22} + ca_{32})\mathbf{f}_2 \\
&\quad + (aa_{13} + ba_{23} + ca_{33})\mathbf{f}_3 \\
&= a'\mathbf{f}_1 + b'\mathbf{f}_2 + c'\mathbf{f}_3.
\end{aligned}$$

These calculations can be expressed in the usual matrix fashion using column vectors formed from the coordinates:

$$\begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Note that, as when you first encountered matrix transformations, the *columns* are formed by the coordinates of the images of the basis vectors.

M101 and *M203*

We shall have quite a lot to do with two special cases of linear transformations: where the domain and codomain are the same and the dimension is two or three.

In such cases the matrix of the transformation is square and there is a simple test for the existence of the inverse transformation. The transformation has an inverse if, and only if, the determinant of the matrix is non-zero.

Such transformations are called non-singular.

We shall write the determinant of a matrix A as both

$$\det(A) \quad \text{and} \quad |A|.$$

As a reminder, the definition of determinant in the 2×2 case is straightforward: if

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

then

$$\det(A) = ad - bc.$$

For 3×3 matrices the definition is

$$\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{32} \\ a_{13} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \end{vmatrix}.$$

Exercise 2.7 Evaluate the following determinants.

(a) $\begin{vmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 3 & 3 \end{vmatrix}$ (b) $\begin{vmatrix} x & y & z \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$ (c) $\begin{vmatrix} x & 1 & -y \\ y & 1 & y \\ y & 1 & x \end{vmatrix}$

[*Solution on page 15*]

This completes a survey of the ideas with which you are expected to be familiar at the start of the course. Further topics will arise during the course but there is usually some practice available.

Solutions to the exercises

Solution 2.1

Using much the same argument as in the text we have

$$\frac{\partial f}{\partial y} = 0 + x(2y) - (\cos(xy))x = 2xy - x\cos(xy).$$

Solution 2.2

We have

$$\frac{\partial g}{\partial x} = y + 2xz - 2y^2 z,$$
$$\frac{\partial g}{\partial y} = x - 4xyz,$$
$$\frac{\partial g}{\partial z} = x^2 - 2xy^2.$$

Solution 2.3

(a) Since, from the definition,

$$\cosh^2 x = \frac{e^{2x} + 2 + e^{-2x}}{4}$$

with a similar result for $\sinh^2 x$, we have

$$\cosh^2 x - \sinh^2 x = \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4}$$
$$= \frac{4}{4} = 1.$$

(b) The quickest way to do this is to use the result from the first part, working from right to left.

$$\begin{aligned} 1 - \tanh^2 x &= 1 - \frac{\sinh^2 x}{\cosh^2 x} \\ &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2} \\ &= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x. \end{aligned}$$

Solution 2.4

We use the definitions and the fact that

$$\frac{d}{dx}e^x = e^x, \quad \frac{d}{dx}e^{-x} = -e^{-x}.$$

(a) We have

$$\cosh' x = \frac{e^x - e^{-x}}{2} = \sinh x.$$

(b) Here

$$\sinh' x = \frac{e^x - (-e^{-x})}{2} = \cosh x.$$

(c) Finally, using the quotient rule,

$$\begin{aligned} \tanh' x &= \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} \\ &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \\ &= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x. \end{aligned}$$

Solution 2.5

(a) From

$$\mathbf{r} = (1-\lambda)\mathbf{p} + \lambda\mathbf{q}$$

we have

$$\begin{aligned} \mathbf{r} &= (1-\lambda)(1,2,-3) + \lambda(3,1,4) \\ &= (1+2\lambda, 2-\lambda, -3+7\lambda). \end{aligned}$$

(b) Inspecting the final form of $\mathbf{r}$ above shows that

$$\mathbf{r} = (1,2,-3) + \lambda(2,-1,7),$$

so

$$\mathbf{v} = (2,-1,7)$$

is a suitable choice.

Solution 2.6

Forming the dot product of $\mathbf{v}$ with $\mathbf{e}_1$ gives

$$\mathbf{v}\cdot\mathbf{e}_1 = \alpha\mathbf{e}_1\cdot\mathbf{e}_1 + \beta\mathbf{e}_2\cdot\mathbf{e}_1 + \gamma\mathbf{e}_3\cdot\mathbf{e}_1.$$

Of the dot products on the right-hand side only the first is non-zero and that has value 1. Hence

$$\mathbf{v}\cdot\mathbf{e}_1 = \alpha.$$

The other two results follow in the same way by taking the dot product of $\mathbf{v}$ with each of the other two basis vectors.

Solution 2.7

We apply the definition given.

(a) $$\begin{vmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 3 & 3 \end{vmatrix} = 0\begin{vmatrix} 0 & 0 \\ 3 & 3 \end{vmatrix} - 1\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + 0\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = -6$$

(b) $$\begin{vmatrix} x & y & z \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = x\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} - y\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} + z\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -z$$

(c) $$\begin{aligned} \begin{vmatrix} x & 1 & -y \\ y & 1 & y \\ y & 1 & x \end{vmatrix} &= x\begin{vmatrix} 1 & y \\ 1 & x \end{vmatrix} - 1\begin{vmatrix} y & y \\ y & x \end{vmatrix} - y\begin{vmatrix} y & 1 \\ y & 1 \end{vmatrix} \\ &= x(x-y) - 1(yx - y^2) \\ &= x(x-y) - y(x-y) \\ &= (x-y)^2 \end{aligned}$$

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